

Optimal Routing of Energy-aware Vehicles in Networks with Inhomogeneous Charging Nodes

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Abstract—We study the routing problem for vehicles with limited energy through a network of inhomogeneous charging nodes. This is substantially more complicated than the homogeneous node case studied in [1]. We seek to minimize the total elapsed time for vehicles to reach their destinations considering both traveling and recharging times at nodes when the vehicles do not have adequate energy for the entire journey. We study two versions of the problem. In the single vehicle routing problem, we formulate a mixed-integer nonlinear programming (MINLP) problem and show that it can be reduced to a lower dimensionality problem by exploiting properties of an optimal solution. We also obtain a Linear Programming (LP) formulation allowing us to decompose it into two simpler problems yielding near-optimal solutions. For a multi-vehicle problem, where traffic congestion effects are included, we use a similar approach by grouping vehicles into “subflows”. We also provide an alternative flow optimization formulation leading to a computationally simpler problem solution with minimal loss in accuracy. Numerical results are included to illustrate these approaches.

I. INTRODUCTION

The increasing presence of Battery-Powered Vehicles (BPVs), such as Electric Vehicles (EVs), mobile robots and sensors, has given rise to novel issues in classical network routing problems [2]. More generally, when entities in a network are characterized by physical attributes exhibiting a dynamic behavior, this behavior can play an important role in the routing decisions. In the case of BPVs, the physical attribute is *energy* and there are four BPV characteristics which are crucial in routing problems: limited cruising range, long charge times, sparse coverage of charging stations, and the BPV energy recuperation ability [3] which can be exploited. In recent years, the vehicle routing literature has been enriched by work aiming to accommodate these BPV characteristics. For example, by incorporating the recuperation ability of EVs, extensions to general shortest-path algorithms are proposed in [3] that address the energy-optimal routing problem, with further extensions in [4]. Charging times are incorporated into a multi-constrained optimal path planning problem in [5], which aims to minimize the length of an EV’s route and meet constraints on total traveling time, total time delay due to signals, total recharging time and total recharging cost. In [6], algorithms for several routing problems are proposed, including a single-vehicle routing problem with inhomogeneously

priced refueling stations for which a dynamic programming based algorithm is proposed to find a least cost path from source to destination. More recently, an EV Routing Problem with Time Windows and recharging stations (E-VRPTW) was proposed in [7], where controlling recharging times is circumvented by simply forcing vehicles to be always fully recharged. In the Unmanned Autonomous Vehicle (UAV) literature, a UAV routing problem with refueling constraints is considered in [8]. A Mixed-Integer Nonlinear Programming (MINLP) formulation is proposed with a heuristic algorithm to determine feasible solutions.

In [1] we studied the energy-constrained vehicle routing problem in a network of homogeneous charging nodes so that the total elapsed time (traveling time and charging time) is minimized. For a single vehicle, a MINLP problem was formulated and, by deriving properties of the optimal solution, we were able to decompose it into two simple Linear Programming (LP) problems. For a multi-vehicle problem, where traffic congestion effects are included and a system-wide objective is considered, a similar approach was used by grouping vehicles into “subflows”.

In this paper, we deal with the vehicle total traveling time minimization problem in a network containing *inhomogeneous* charging nodes, i.e., charging rates at different nodes are not identical. In fact, depending on an outlet’s voltage and current, charging an EV battery could take anywhere from minutes to hours and the Society of Automotive Engineering (SAE) classifies charging stations into three categories [9], [10], [11] as shown in Tab. I. Thus, charging rates and times are highly dependent on the charging station class and clearly affect the solution of our optimization problem. As in [1], we view this

TABLE I
CLASSIFICATION OF CHARGING STATIONS [10]

Charge Method	Nominal Supply Voltage(volts)	Max. Current (Amps)	Miles per every hour charging
AC Level 1	120 VAC, 1-phase	12 A	< 5
AC Level 2	208-240 VAC, 1-phase	32 A	up to 62
DC Charging	300 - 460VDC	400 A Max.	up to 300

as a network routing problem where vehicles control not only their routes but also amounts to recharge at various nodes in the network.

The contributions of this paper are as follows. For the single

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energy-aware vehicle routing problem, due to the inhomogeneity in charging nodes, we can no longer reduce the original problem to a simple LP as in [1]. However, we can still prove certain optimality properties allowing us to reduce the dimensionality of the original problem. Further, by adopting a locally optimal charging policy, we derive an LP formulation through which near-optimal solutions are obtained. We note that the main difference between this single-vehicle problem and the one considered in [6] is that we aim to minimize the total elapsed time over a path, as opposed to a fueling cost, and must, therefore, include two parameters per network arc: energy consumption and traveling time. We then study a multi-vehicle energy-aware routing problem, where a traffic flow model is used to incorporate congestion effects. Similar to [1], by grouping vehicles into “subflows” we are able to reduce the complexity of the original problem, although we can no longer obtain an LP formulation. Moreover, we provide an alternative flow-based formulation which reduces the computational complexity of the original MINLP problem by orders of magnitude with numerical results showing little loss in optimality.

The structure of the paper is as follows. In Section II, we address the single-vehicle routing problem in a network with inhomogeneous charging nodes and identify properties which lead to its simplification. In Section III, the multi-vehicle routing problem is formulated, first as a MINLP and then as an alternative flow optimization problem. Simulation examples are included illustrating our approach and providing insights on the relationship between recharging speed and optimal routes. Conclusions and further research directions are outlined in Section IV.

II. SINGLE VEHICLE ROUTING

We assume that a network is defined as a directed graph $G = (\mathcal{N}, \mathcal{A})$ with $\mathcal{N} = \{1, \dots, n\}$ and $|\mathcal{A}| = m$. Node $i \in \mathcal{N}/\{n\}$ represents a charging station and $(i, j) \in \mathcal{A}$ is an arc connecting node i to j (we assume for simplicity that all nodes have a charging capability, although this is not necessary). We also define $I(i)$ and $O(i)$ to be the set of start nodes (respectively, end nodes) of arcs that are incoming to (respectively, outgoing from) node i , that is, $I(i) = \{j \in \mathcal{N} | (j, i) \in \mathcal{A}\}$ and $O(i) = \{j \in \mathcal{N} | (i, j) \in \mathcal{A}\}$.

First we deal with a single-origin-single-destination vehicle routing problem in a network of inhomogeneous charging stations. Nodes 1 and n respectively are defined to be the origin and destination. For each arc $(i, j) \in \mathcal{A}$, there are two cost parameters: the required traveling time τ_{ij} and the energy consumption e_{ij} . Note that $\tau_{ij} > 0$ (if nodes i and j are not connected, then $\tau_{ij} = \infty$), whereas e_{ij} is allowed to be negative due to a BPV’s potential energy recuperation effect [3]. Letting the vehicle’s charge capacity be B , we assume that $e_{ij} < B$ for all $(i, j) \in \mathcal{A}$. Since we are considering a single vehicle’s behavior, we assume that it will not affect the overall network’s traffic state, therefore, τ_{ij} and e_{ij} are fixed depending on given traffic conditions at the time the single-vehicle routing problem is solved. Clearly, this cannot apply to the multi-vehicle case in the next section, where the

decisions of multiple vehicle routes affect traffic conditions, thus influencing traveling times and energy consumption. Since the BPV has limited battery energy it may not be able to reach the destination without recharging. Thus, recharging amounts at charging nodes $i \in \mathcal{N}$ are also decision variables.

We denote the selection of arc (i, j) and energy recharging amount at node i by $x_{ij} \in \{0, 1\}$, $i, j \in \mathcal{N}$ and $r_i \geq 0$, $i \in \mathcal{N}/\{n\}$, respectively. We also define g_i as the charging time per unit of energy for charging node i , i.e., the reciprocal of a charging rate for each node. Without loss of generality we assume $g_n = 0$. Moreover, we use E_i to represent the vehicle’s residual battery energy at node i . Then, for all E_j , $j \in O(i)$, we have:

$$E_j = \begin{cases} E_i + r_i - e_{ij} & \text{if } x_{ij} = 1 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

which can also be expressed as

$$E_j = \sum_{i \in I(j)} (E_i + r_i - e_{ij})x_{ij}, \quad x_{ij} \in \{0, 1\}$$

The single vehicle’s objective is to determine a path from 1 to n , as well as recharging amounts, so as to minimize the total elapsed time to reach the destination. We formulate this as a Mixed Integer Nonlinear Programming (MINLP) problem:

$$\min_{x_{ij}, r_i, i, j \in \mathcal{N}} \sum_{i=1}^n \sum_{j=1}^n \tau_{ij} x_{ij} + \sum_{i=1}^n \sum_{j=1}^n r_i g_i x_{ij} \quad (2)$$

$$\text{s.t.} \quad \sum_{j \in O(i)} x_{ij} - \sum_{j \in I(i)} x_{ji} = b_i, \quad \text{for each } i \in \mathcal{N} \quad (3)$$

$$b_1 = 1, b_n = -1, b_i = 0, \text{ for } i \neq 1, n \quad (4)$$

$$E_j = \sum_{i \in I(j)} (E_i + r_i - e_{ij})x_{ij}, \text{ for } j = 2, \dots, n \quad (5)$$

$$0 \leq E_i \leq B, \quad E_1 \text{ given, for each } i \in \mathcal{N} \quad (6)$$

$$x_{ij} \in \{0, 1\}, \quad r_i \geq 0 \quad (7)$$

We will refer to this problem as **P1**. The constraints (3)-(4) stand for the flow conservation, which implies that only one path starting from node i can be selected, i.e., $\sum_{j \in O(i)} x_{ij} \leq 1$. It is easy to check that this also implies $x_{ij} \leq 1$ for all i, j since $b_1 = 1$, $I(1) = \emptyset$. Constraint (5) represents the vehicle’s energy dynamics where the only nonlinearity in this formulation appears. Finally, (6) indicates that the vehicle cannot run out of energy before reaching a node or exceed a given capacity B . All other parameters are predetermined according to the network topology. A crucial difference between **P1** and the MINLP introduced in [1] is that here the charging rates g_i in (2) are node-dependent.

A. Properties

Rather than directly tackling the MINLP problem, we derive some key properties of an optimal solution which will enable us to reduce **P1** to a lower-dimension problem. In particular, there are $m + 2(n - 1)$ decision variables in **P1** (because of the E_j variables in (5)), which we will show how to reduce to $m + (n - 1)$. The main difficulty in this problem lies in the coupling of the decision variables x_{ij} and r_i in (5) and the following lemma will enable us to eliminate r_i from (2).

Lemma 1: Given (2)-(7), an optimal solution $\{x_{ij}, r_i\}$, $i, j \in$

\mathcal{N} satisfies:

$$\sum_{i=1}^n \sum_{j=1}^n (r_i x_{ij} - e_{ij} x_{ij}) g_i = \sum_{i=1}^n \sum_{j=1}^n (E_j - E_i) g_i x_{ij} \quad (8)$$

$$= \sum_{i=1}^n \sum_{j=1}^n E_j (g_i - g_j) x_{ij} - E_1 g_1 \quad (9)$$

Proof: Multiplying both sides of (1) by g_i gives:

$$E_j g_i = \begin{cases} (E_i + r_i - e_{ij}) g_i & \text{if } x_{ij} = 1, \\ 0 & \text{otherwise.} \end{cases}$$

which can be expressed as $\sum_{i \in I(j)} E_j g_i x_{ij} = \sum_{i \in I(j)} (E_i + r_i - e_{ij}) g_i x_{ij}$. Summing both sides over $j = 2, \dots, n$ and rearranging yields:

$$\begin{aligned} \sum_{j=2}^n \sum_{i \in I(j)} E_j g_i x_{ij} - \sum_{j=2}^n \sum_{i \in I(j)} E_i g_i x_{ij} \\ = \sum_{j=2}^n \sum_{i \in I(j)} (r_i - e_{ij}) g_i x_{ij} \end{aligned}$$

Based on (1), $E_i = 0$ for all nodes which are not in the selected path. Thus we can rewrite the equation above as

$$\sum_{i=1}^n \sum_{j=1}^n (r_i x_{ij} - e_{ij} x_{ij}) g_i = \sum_{i=1}^n \sum_{j=1}^n (E_j - E_i) g_i x_{ij}$$

which establishes (8). Finally, (9) follows by observing that if P is an optimal path we can re-index nodes so that $P = \{1, \dots, n\}$ with $g_n = 0$. Thus, we have $\sum_{i=1}^n \sum_{j=1}^n E_i g_i x_{ij} = E_1 g_1 + \dots + E_{n-1} g_{n-1}$ which can also be written as $E_1 g_1 + \sum_{i=2}^n \sum_{j=2}^n E_j g_j x_{ij}$ where $x_{ij} = 0$ for all (i, j) not in the optimal path. Therefore,

$$\sum_{i=1}^n \sum_{j=1}^n (E_j - E_i) g_i x_{ij} = \sum_{i=1}^n \sum_{j=1}^n E_j (g_i - g_j) x_{ij} - E_1 g_1$$

which proves (9). ■

Lemma 2: If $\sum_i r_i^* > 0$ in the optimal routing policy, then $E_n^* = 0$.

Proof: This is the same as the homogeneous charging node case; see Lemma 2 in [1].

In view of Lemma 1, we can replace $\sum_{i=1}^n \sum_{j=1}^n r_i g_i x_{ij}$ in (2) and eliminate the presence of r_i , $i = 2, \dots, n-1$, from the objective function and the constraints. This results in the following MINLP problem referred to as **P2**:

$$\min_{\substack{x_{ij}, E_i \\ i, j \in \mathcal{N}}} \sum_{i=1}^n \sum_{j=1}^n (\tau_{ij} x_{ij} + e_{ij} g_i x_{ij} + E_j (g_i - g_j) x_{ij}) - E_1 g_1 \quad (10)$$

$$s.t. \quad \sum_{j \in O(i)} x_{ij} - \sum_{j \in I(i)} x_{ji} = b_i \quad (11)$$

$$b_1 = 1, b_n = -1, b_i = 0 \quad \text{for } i \neq 1, n \quad (12)$$

$$0 \leq E_j - (E_i - e_{ij}) x_{ij} \leq B \quad \forall i, j \in \mathcal{N} \quad (13)$$

$$0 \leq E_i \leq B, E_1 \text{ given}, \forall i \in \mathcal{N} \quad (14)$$

$$x_{ij} \in \{0, 1\} \quad (15)$$

This new formulation has only $m + (n-1)$ decision variables compared to $m + 2(n-1)$ in **P1**. Constraint (13) is derived from (5). Assuming $x_{ij} = 1$, i.e. arc (i, j) is part of the optimal

path, we can recover $r_i = E_j - E_i + e_{ij}$ and Constraint (13) is added to prevent any vehicle from exceeding its capacity B in an optimal path. Solving this problem gives both an optimal path and residual battery energy at each node.

Although **P2** has fewer decision variables, it is still a MINLP which is hard to solve for large networks. Specifically, the CPU time is highly dependent on the number of nodes and arcs in the network. In what follows we introduce a *locally optimal* charging policy, leading to a simpler problem, by arguing as follows. Looking at (10), the term $\sum_{i=1}^n \sum_{j=1}^n E_j (g_i - g_j) x_{ij}$ is minimized by selecting each E_j depending on the sign of $(g_i - g_j)$:

Case 1: $g_i - g_j < 0$, i.e., node i has a faster charging rate than node j . Therefore, E_j should get its maximum possible value, which is $B - e_{ij}$. This implies that the vehicle must be maximally charged at node i .

Case 2: $g_i - g_j \geq 0$, i.e., node j has a faster or same charging rate as node i . In this case, E_j should get its minimum value $E_j = 0$. This implies that the vehicle should get the minimum charge needed at node i in order to reach node j .

We define π_C to be the charging policy specified as above and note that it does not guarantee the global optimality of E_i thus selected in (10) which can easily be checked by a counterexample. However, it allows us to decompose the optimal routing problem from the optimal charging problem. If, in addition, we consider only solutions for which the vehicle is recharged at least once (otherwise, the vehicle is not energy-constrained and the problem is of limited interest), we can obtain the following result.

Theorem 1: If $\sum_i r_i^* > 0$ (i.e. the vehicle has to be recharged at least once), then under charging policy π_C , the solution x_{ij}^* , $i, j \in \mathcal{N}$, of the original problem (2) can be determined by solving the LP problem:

$$\min_{x_{ij}, i, j \in \mathcal{N}} \sum_{i=1}^n \sum_{j=1}^n (\tau_{ij} + e_{ij} g_i + K(g_i - g_j)) x_{ij} \quad (16)$$

$$K = \begin{cases} B - e_{ij} & \text{if } g_i < g_j, \\ 0 & \text{otherwise.} \end{cases} \quad (17)$$

$$s.t. \quad \sum_{j \in O(i)} x_{ij} - \sum_{j \in I(i)} x_{ji} = b_i \quad (18)$$

$$b_1 = 1, b_n = -1, b_i = 0 \quad \text{for } i \neq 1, n \quad (19)$$

$$0 \leq x_{ij} \leq 1 \quad (20)$$

Proof: Applying charging policy π_C in (10) we can change objective function to $\sum_{i=1}^n \sum_{j=1}^n (\tau_{ij} + e_{ij} g_i + K(g_i - g_j)) x_{ij} - E_1 g_1$ where K is as in (17). Therefore, x_{ij}^* can be determined by the following problem:

$$\min_{x_{ij}, i, j \in \mathcal{N}} \sum_{i=1}^n \sum_{j=1}^n (\tau_{ij} + e_{ij} g_i + K(g_i - g_j)) x_{ij} - E_1 g_1$$

$$K = \begin{cases} B - e_{ij} & \text{if } g_i < g_j, \\ 0 & \text{otherwise.} \end{cases}$$

$$s.t. \quad \sum_{j \in O(i)} x_{ij} - \sum_{j \in I(i)} x_{ji} = b_i, \quad \text{for each } i \in \mathcal{N}$$

$$b_1 = 1, b_n = -1, b_i = 0, \quad \text{for } i \neq 1, n$$

$$x_{ij} \in \{0, 1\}$$

which is a typical shortest path problem formulation. Moreover, according to the property of minimum cost flow problems [12], the above integer programming problem is equivalent to the LP with the integer restriction on x_{ij} relaxed. Finally, since E_1 and g_1 are given, the problem reduces to (16), which proves the theorem. ■

Remark 1. If $g_i = g_j$ for all i, j in (16), the problem reduces to the homogeneous charging node case studied in [1] with the same optimal LP formulation as in Theorem 1. With $g_i \neq g_j$ however, the LP formulation cannot guarantee global optimality, although the routes obtained through Theorem 1 may indeed be optimal (see Section II.C), in which case the optimal charging amounts are obtained as described next.

B. Determination of optimal recharging amounts r_i^*

Once we determine an optimal route P , it is relatively easy to find a feasible solution for r_i , $i \in P$, to satisfy the constraint (5) and minimize the total charging time on the selected path. It is obvious that the optimal charging amounts r_i^* are non-unique in general. Without loss of generality we re-index nodes so that we may write $P = \{1, \dots, n\}$. Then, the problem resulting in an optimal charging policy is

$$\begin{aligned} \min_{r_i, i \in P} \quad & \sum_{i \in P} g_i r_i \\ \text{s.t.} \quad & E_{i+1} = E_i + r_i - e_{i,i+1} \\ & 0 \leq E_i \leq B, \quad E_1 \text{ given} \\ & r_i \geq 0 \text{ for all } i \in \mathcal{N} \end{aligned} \quad (21)$$

This is an LP where E_i and r_i are decision variables. Unlike the homogeneous charging node problem in [1] where the objective function includes charging prices p_i associated with nodes, i.e., $\sum_{i \in P} p_i r_i$, this is not the case here, since there is a tradeoff between selecting faster-charging nodes and possible higher costs at such nodes. However, the advantage of the decoupling approach is that if an optimal path is determined, an additional cost minimization problem can be formulated to determine optimal charging times at nodes on this path.

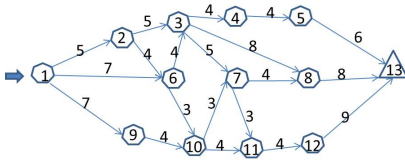


Fig. 1. 13-node network with inhomogeneous charging nodes

C. Numerical Example

We consider a 13-node network as shown in Fig.1 where the travel distance of each arc is as shown. For simplicity here we assume $\tau_{ij} = e_{ij} = d_{ij}$ and solve the problem for different configurations of charging stations in the network. The optimal paths obtained by solving different formulations are shown in Tab.II where $G = [g_1, g_2, \dots, g_{n-1}]$. We can see that all formulations result in the *same* optimal path while their computational complexities are drastically different

(from around 250 sec for **P1** to less than 2 sec for the LP formulation). Once the optimal path is determined, we can easily solve (21) to determine optimal charging amounts as well.

TABLE II
OPTIMAL PATHS OBTAINED BY SOLVING PROBLEMS **P-1**, **P-2** AND LP FOR DIFFERENT CHARGING NODE CONFIGURATIONS

G	Problem	Path
[1 0.2 1 0.1 1 0.2 0.1 1 1 1 1 1]	P1& P2: LP:	1→2→3→4→5→13 1→2→3→4→5→13
[1 1 1 1 1 1 0.2 0.1 1 0.1 1 1]	P1& P2: LP:	1→6→10→7→8→13 1→6→10→7→8→13
[1 1 1 1 1 1 1 1 1 0.1 1 1]	P1& P2: LP:	1→6→10→7→8→13 1→6→10→7→8→13
[1 1 1 1 1 1 1 1 1 1 1 1]	P1& P2: LP:	1→2→3→4→5→13 1→2→3→4→5→13
[1 1 1 1 1 1 1 0.1 0.1 1 1 1]	P1& P2: LP:	1→9→10→7→8→13 1→9→10→7→8→13

III. MULTIPLE VEHICLE ROUTING

We now investigate the multi-vehicle routing problem in a network with inhomogeneous charging nodes, where we seek to optimize a *system-wide objective* by routing and charging vehicles. The main technical difficulty in this case is that we need to consider the influence of traffic congestion on both traveling time and energy consumption. A second difficulty is that of implementing an optimal routing policy. In the case of a centrally controlled system consisting of mobile robots, sensors or any type of autonomous vehicles this can be accomplished through appropriately communicated commands. In the case of vehicles with individual drivers, implementation requires signaling mechanisms and possibly incentive structures to enforce desired routes assigned to vehicles, bringing up a number of additional research issues. In the sequel, we limit ourselves to resolving the first difficulty before addressing implementation challenges.

If we proceed as in the single vehicle case, i.e., determining a path selection through x_{ij}^k , $i, j \in \mathcal{N}$, and recharging amounts r_i^k , $i \in \mathcal{N}/\{n\}$ for all vehicles $k = 1, \dots, K$, for some K , then the dimensionality of the solution space is prohibitive. Moreover, the inclusion of traffic congestion effects introduces additional nonlinearities in the dependence of the travel time τ_{ij} and energy consumption e_{ij} on the traffic flow through arc (i, j) , which now depend on $x_{ij}^1, \dots, x_{ij}^K$. Instead, as in [1], we proceed by grouping subsets of vehicles into N “subflows” where N may be selected to render the problem manageable (we will discuss the effect of N in Section III.C).

Let all vehicles enter the network at node 1 and let R denote the rate of vehicles arriving at this node. Viewing vehicles as defining a *flow*, we divide them into N *subflows* each of which may be selected so as to include the same type of homogeneous vehicles (e.g., large vehicles vs smaller ones or vehicles with the same initial energy). Thus, all vehicles in the same subflow follow the same routing and recharging decisions so that we only consider control at the subflow level rather than individual vehicles. Note that asymptotically, as $N \rightarrow \infty$, we can recover routing at the individual vehicle

level. Clearly, not all vehicles in our system are BPVs, therefore, not part of our optimization process. These can be treated as uncontrollable interfering traffic for our purposes and can be readily accommodated in our analysis, as long as their flow rates are known. However, for simplicity, we will assume here that every arriving vehicle is a BPV and joins a subflow. Our objective is to determine optimal routes and energy recharging amounts for each vehicle subflow so as to minimize the total elapsed time of these flows from origin to destination. The decision variables are $x_{ij}^k \in \{0, 1\}$ and r_i^k for all arcs (i, j) and subflows $k = 1, \dots, N$. Given traffic congestion effects, the time and energy consumption on each arc depends on the values of x_{ij}^k and the fraction of the total flow rate R associated with each subflow k ; the simplest such flow allocation is one where each subflow is assigned R/N . Let $\mathbf{x}_{ij} = (x_{ij}^1, \dots, x_{ij}^N)^T$ and $\mathbf{r}_i = (r_i^1, \dots, r_i^N)^T$. Then, we denote the traveling time and corresponding energy consumption of the k th vehicle subflow on arc (i, j) by $\tau_{ij}^k(\mathbf{x}_{ij})$ and $e_{ij}^k(\mathbf{x}_{ij})$ respectively. As already mentioned, $\tau_{ij}^k(\mathbf{x}_{ij})$ and $e_{ij}^k(\mathbf{x}_{ij})$ can also incorporate the influence of uncontrollable (non-BPV) vehicle flows, which can be treated as parameters in these functions. Similar to the single vehicle case, we use E_i^k to represent the residual energy of subflow k at node i , given by the aggregated residual energy of all vehicles in the subflow. If the subflow does not go through node i , then $E_i^k = 0$. Similar to [1], the problem is formulated as follows:

$$\min_{\mathbf{x}_{ij}, \mathbf{r}_i, i, j \in \mathcal{N}} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^N (\tau_{ij}^k(\mathbf{x}_{ij}) + r_i^k g_i x_{ij}^k) \quad (22)$$

$$\text{s.t.} \quad \sum_{j \in O(i)} x_{ij}^k - \sum_{j \in I(i)} x_{ji}^k = b_i, \quad \text{for each } i \in \mathcal{N} \quad (23)$$

$$b_1 = 1, b_n = -1, b_i = 0, \quad \text{for } i \neq 1, n \quad (24)$$

$$E_j^k = \sum_{i \in I(j)} (E_i^k + r_i^k - e_{ij}^k(\mathbf{x}_{ij})) x_{ij}^k, \quad j = 2, \dots, n \quad (25)$$

$$E_1^k \text{ is given, } E_i^k \geq 0, \quad \text{for each } i \in \mathcal{N} \quad (26)$$

$$x_{ij}^k \in \{0, 1\}, \quad r_i^k \geq 0 \quad (27)$$

We will refer to this problem **P3**. The difference from the MINLP formulated in [1] is that we consider different charging rates g_i in the objective function. **P3** contains $N(m+2(n-1))$ decision variables and is difficult to solve. However, as in the single-vehicle case, we are able to establish some properties allowing us to simplify it.

A. Properties

Even though the term $\tau_{ij}^k(\mathbf{x}_{ij})$ in the objective function is no longer linear in general, for each subflow k the constraints (23)-(27) are still similar to the single-vehicle case. Consequently, we can derive similar useful properties in the form of the following lemmas (proofs are very similar to those of the single-vehicle case and are omitted).

Lemma 3: An optimal solution $\{\mathbf{x}_{ij}, \mathbf{r}_i\}$, $i, j \in \mathcal{N}$ satisfies:

$$\begin{aligned} & \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^N r_i^k g_i x_{ij}^k - \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^N e_{ij}^k(\mathbf{x}_{ij}) g_i x_{ij}^k \\ &= \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^N (E_j^k - E_i^k) g_i x_{ij}^k \end{aligned} \quad (28)$$

$$= \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^N E_j^k (g_i - g_j) x_{ij}^k - \sum_{k=1}^N E_1^k g_1 \quad (29)$$

Lemma 4: If $\sum_i r_i^{k*} > 0$ in the optimal routing policy, then $E_n^{k*} = 0$ for $k = 1, \dots, N$.

In view of Lemma 3, we can replace $\sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^N r_i^k g_i x_{ij}^k$ in (22) through (29) and eliminate r_i^k , $i = 1, \dots, n-1$, $k = 1, \dots, N$, from the objective function (22). The term $\sum_{k=1}^N E_1^k g_1$ is also removed because it has a fixed value. Thus, we introduce a new MINLP formulation to determine x_{ij}^{k*} and E_i^{k*} for all $i, j \in \mathcal{N}$ and $k = 1, \dots, N$ as follows:

$$\min_{\substack{x_{ij}^k, E_i^k \\ i, j \in \mathcal{N}}} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^N (\tau_{ij}^k(\mathbf{x}_{ij}) + (e_{ij}^k(\mathbf{x}_{ij}) g_i + E_j^k (g_i - g_j)) x_{ij}^k) \quad (30)$$

$$\text{s.t.} \quad \sum_{j \in O(i)} x_{ij}^k - \sum_{j \in I(i)} x_{ji}^k = b_i \quad (31)$$

$$b_1 = 1, b_n = -1, b_i = 0 \quad \text{for } i \neq 1, n$$

$$0 \leq E_j^k - (E_i^k - e_{ij}^k(\mathbf{x}_{ij})) x_{ij}^k \leq B^k \quad \text{for each } (i, j) \in A \quad (32)$$

$$E_1^k \text{ is given, } E_i^k \geq 0, \quad \text{for each } i \in \mathcal{N} \quad (33)$$

$$x_{ij}^k \in \{0, 1\} \quad (34)$$

We call this problem **P4**. Note that inequality (32) is derived from (25). Assuming $x_{ij}^k = 1$, i.e., arc (i, j) is part of the optimal path for the k th subflow, $r_i^k = e_{ij}^k(\mathbf{x}_{ij}) + E_j^k - E_i^k$. Thus, (32) ensures the optimal solution E_i^{k*} results in a feasible charging amount for the k th subflow, $0 \leq r_i^k \leq B^k$ where B^k is the maximum charging amount k th subflow can get. This value should be predetermined for each subflow based on the vehicle types and the fraction of total inflow in it. Similar to **P2** in the single-vehicle case, once we determine E_i^{k*} we can simply calculate optimal charging amounts using (25). Although **P4** has $N(m + (n-1))$ decision variables, which is fewer than **P3**, its complexity still highly depends on the network size and number of subflows. Similar to the charging policy π_C used in Theorem 1, we introduce a charging policy by arguing as follows. Looking at (30), the term $\sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^N E_j^k (g_i - g_j) x_{ij}^k$ is minimized by selecting each E_j^k depending on the sign of $(g_i - g_j)$:

Case 1: $g_i < g_j$, i.e., node i has faster charging rate than node j . Therefore, E_j^k should get its maximum value, i.e., the k th subflow should get its maximum charge at node i .

Case 2: $g_i \geq g_j$, i.e., the charging rate of node j is greater than or equal to node i . Therefore, E_j^k should get its minimum value of 0. This implies that the k th subflow should get the minimum charge needed at node i in order to reach node j . Applying this policy in (30) and changing the objective function accordingly we introduce problem **P5** as follows:

$$\min_{x_{ij}^k} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^N [\tau_{ij}^k(\mathbf{x}_{ij}) + (e_{ij}^k(\mathbf{x}_{ij}) g_i + K (g_i - g_j)) x_{ij}^k] \quad (35)$$

$$K = \begin{cases} B^k - e_{ij}^k(\mathbf{x}_{ij}) & \text{if } g_i < g_j, \\ 0 & \text{otherwise} \end{cases} \quad (36)$$

$$\text{s.t.} \quad \sum_{j \in O(i)} x_{ij}^k - \sum_{j \in I(i)} x_{ji}^k = b_i \quad (37)$$

$$b_1 = 1, b_n = -1, b_i = 0 \quad \text{for } i \neq 1, n$$

$$x_{ij}^k \in \{0, 1\} \quad (38)$$

Unlike the single-vehicle case, the objective function is no longer necessarily linear in x_{ij}^k , therefore, (35) cannot be further simplified into an LP problem as in Theorem 1. The computational effort required to solve this problem with Nm decision variables, depends on the dimensionality of the network and the number of subflows. Nonetheless, from the transformed formulation above, we are still able to separate the determination of routing variables x_{ij}^k from recharging amounts r_i^k . Similar to the single-vehicle case, once the routes are determined, we can obtain r_i^k satisfying the energy constraints (25)-(26) while minimizing $\sum_{k=1}^N \sum_{i \in P^k} r_i^k g_i$ where P^k is the optimal path of the k th subflow. Next, we present an alternative formulation of (22)-(27) leading to a computationally simpler solution approach.

Remark 2. If $g_i = g_j$ for all i, j in (35), the problem reduces to the homogeneous charging node case with the exact same MINLP formulation as in [1] for obtaining an optimal path. However, **P5** cannot guarantee an optimal solution because of the locally optimal charging policy π_C which may not be feasible in a globally optimal solution $(x_{ij}^{k*}, E_{ij}^{k*})$.

B. Flow control formulation

We begin by relaxing the binary variables in (27) by letting $0 \leq x_{ij}^k \leq 1$. Thus, we switch our attention from determining a single path for any subflow k to several possible paths by treating x_{ij}^k as the normalized vehicle flow on arc (i, j) for the k th subflow. This is in line with many network routing algorithms in which fractions x_{ij} of entities are routed from a node i to a neighboring node j using appropriate schemes ensuring that, in the long term, the fraction of entities routed on (i, j) is indeed x_{ij} . Following this relaxation, the objective function in (22) is changed to:

$$\min_{\mathbf{x}_{ij}, \mathbf{r}_i, i, j \in \mathcal{N}} \quad \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^N \tau_{ij}^k(\mathbf{x}_{ij}) + \sum_{i=1}^n \sum_{k=1}^N r_i^k g_i$$

Moreover, the energy constraint (25) needs to be adjusted accordingly. Let E_{ij}^k represent the fraction of residual energy of subflow k associated with the x_{ij}^k portion of the vehicle flow exiting node i . Therefore, constraint (26) becomes $E_{ij}^k \geq 0$. We can now capture the relationship between the energy associated with subflow k and the vehicle flow as follows:

$$\left[\sum_{h \in I(i)} (E_{hi}^k - e_{hi}^k(\mathbf{x}_{ij})) + r_i^k \right] \cdot \frac{x_{ij}^k}{\sum_{h \in I(i)} x_{hi}^k} = E_{ij}^k \quad (39)$$

$$\frac{E_{ij}^k}{\sum_{j \in O(i)} E_{ij}^k} = \frac{x_{ij}^k}{\sum_{j \in O(i)} x_{ij}^k} \quad (40)$$

In (39), the energy values of different vehicle flows entering node i are aggregated and the energy corresponding to each portion exiting a node, $E_{ij}^k, j \in O(i)$, is proportional to the corresponding fraction of vehicle flows, as expressed in (40). Clearly, this aggregation of energy leads to an approximation,

since one specific vehicle flow may need to be recharged in order to reach the next node in its path, whereas another might have enough energy without being recharged. This approximation foregoes controlling recharging amounts at the individual vehicle level and leads to approximate solutions of the original problem (22)-(27). Several numerically based comparisons are provided in the next section showing little or no loss of optimality relative to the solution of (22). Adopting this formulation with $x_{ij}^k \in [0, 1]$ instead of $x_{ij}^k \in \{0, 1\}$, we obtain the following simpler nonlinear programming problem (NLP):

$$\min_{\mathbf{x}_{ij}, \mathbf{r}_i, i, j \in \mathcal{N}} \quad \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^N \tau_{ij}^k(\mathbf{x}_{ij}) + \sum_{i=1}^n \sum_{k=1}^N r_i^k g_i \quad (41)$$

$$\text{s.t.} \quad \sum_{j \in O(i)} x_{ij}^k - \sum_{j \in I(i)} x_{ji}^k = b_i, \quad \text{for each } i \in \mathcal{N} \quad (42)$$

$$b_1 = 1, b_n = -1, b_i = 0, \quad \text{for } i \neq 1, n$$

$$\left[\sum_{h \in I(i)} (E_{hi}^k - e_{hi}^k(\mathbf{x}_{ij})) + r_i^k \right] \cdot \frac{x_{ij}^k}{\sum_{h \in I(i)} x_{hi}^k} = E_{ij}^k \quad (43)$$

$$\frac{E_{ij}^k}{\sum_{j \in O(i)} E_{ij}^k} = \frac{x_{ij}^k}{\sum_{j \in O(i)} x_{ij}^k} \quad (44)$$

$$E_{ij}^k \geq 0, \quad (45)$$

$$0 \leq x_{ij}^k \leq 1, \quad r_i^k \geq 0 \quad (46)$$

As in our previous analysis, we are able to eliminate \mathbf{r}_i from the objective function in (41) as follows.

Lemma 5: For each subflow $k = 1, \dots, N$,

$$\begin{aligned} \sum_{i=1}^n r_i^k g_i &= \sum_{i=1}^n \sum_{j=1}^n e_{ij}^k(\mathbf{x}_{ij}) g_i + \sum_{i=1}^n \sum_{j \in O(i)} E_{ij}^k g_i - \sum_{i=1}^n \sum_{h \in I(i)} E_{hi}^k g_i \\ &= \sum_{i=1}^n \sum_{j=1}^n e_{ij}^k(\mathbf{x}_{ij}) g_i + \sum_{i=1}^n \sum_{j \in O(i)} E_{ij}^k (g_i - g_j) \end{aligned}$$

Proof: Multiplying (43) by g_i and summing over all $i = 1, \dots, n$, then using (42) and (44) proves the lemma. ■

Using Lemma 5 we change the objective function (41) to:

$$\sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^N (\tau_{ij}^k(\mathbf{x}_{ij}) + e_{ij}^k(\mathbf{x}_{ij}) g_i) + \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^N E_{ij}^k (g_i - g_j) \quad (47)$$

Once again, we adopt a charging policy π_C as follows:

Case 1: If $g_i < g_j$, then E_{ij}^k gets its maximum value $(B^k - e_{ij}^k(\mathbf{x}_{ij})) x_{ij}^k$.

Case 2: If $g_i \geq g_j$, then E_{ij}^k gets its minimum value 0.

Applying this policy in (47) we can transform the objective function (41) to (48) and determine near-optimal routes x_{ij}^{k*} by solving the following NLP:

$$\min_{\mathbf{x}_{ij}, i, j \in \mathcal{N}} \quad \sum_{k=1}^N \sum_{i=1}^n \sum_{j=1}^n [\tau_{ij}^k(\mathbf{x}_{ij}) + e_{ij}^k(\mathbf{x}_{ij}) g_i + K(g_i - g_j)] \quad (48)$$

$$\begin{aligned}
K &= \begin{cases} (B^k - e_{ij}^k(\mathbf{x}_{ij}))x_{ij}^k & \text{if } g_i < g_j, \\ 0 & \text{otherwise} \end{cases} \\
\text{s.t. } \sum_{j \in O(i)} x_{ij}^k - \sum_{j \in I(i)} x_{ji}^k &= b_i, \quad \text{for each } i \in \mathcal{N} \\
b_1 &= 1, b_n = -1, b_i = 0, \quad \text{for } i \neq 1, n \\
0 &\leq x_{ij}^k \leq 1
\end{aligned}$$

Once again, there is no guarantee of global optimality. The values of r_i^k , $i = 1, \dots, n$, $k = 1, \dots, N$, can then be determined so as to satisfy the energy constraints (43)-(45), and minimizing $\sum_{k=1}^N \sum_{i \in P^k} r_i^k g_i$. Finally, if $g_i = g_j$ for all i, j in (48), the problem reduces to the homogeneous charging node case with the same exact NLP flow control formulation as in [1].

C. Numerical Examples

We consider a specific example which includes traffic congestion and energy consumption functions. The relationship between the speed and density of a vehicle flow is typically estimated as follows (see [13]):

$$v(k(t)) = v_f \left(1 - \left(\frac{k(t)}{k_{jam}} \right)^p \right)^q \quad (49)$$

where v_f is the reference speed on the road without traffic, $k(t)$ represents the density of vehicles on the road at time t and k_{jam} the saturated density for a traffic jam. The parameters p and q are empirically identified for actual traffic flows. In our multi-vehicle routing problem, we are interested in the relationship between the density of the vehicle flow and traveling time on an arc (i, j) , i.e., $\tau_{ij}^k(\mathbf{x}_{ij})$. Given a network topology (i.e., a road map), the distances d_{ij} between nodes are known. Moreover, we do not include uncontrollable vehicle flows in our example for simplicity. In our approach, we need to identify N subflows and we do so by evenly dividing the entire vehicle inflow into N subflows, each of which has R/N vehicles per unit time. Thus, k_{jam} in this case can be set as N , implying that we do not want all vehicles to go through the same path, hence the the arc (i, j) density is $\sum_k x_{ij}^k$. Therefore, the time subflow k spends on arc (i, j) becomes

$$\tau_{ij}^k(\mathbf{x}_{ij}) = (d_{ij} \cdot x_{ij}^k \cdot \frac{R}{N}) (v_f (1 - (\frac{\sum_k x_{ij}^k}{N})^p)^q)^{-1}$$

As for $e_{ij}^k(\mathbf{x}_{ij})$, we assume the energy consumption rates of subflows on arc (i, j) are all identical, proportional to the distance between nodes i and j , giving

$$e_{ij}^k(\mathbf{x}_{ij}) = e \cdot d_{ij} \cdot \frac{R}{N}$$

In order to verify the accuracy of different formulations, we numerically solve the optimal and near-optimal problems **P4** and **P5**. For the latter, (35) becomes:

$$\begin{aligned}
\min_{\substack{x_{ij}^k \\ i, j \in \mathcal{N}}} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^N & \left[\frac{d_{ij} x_{ij}^k \frac{R}{N}}{v_f (1 - (\frac{\sum_k x_{ij}^k}{N})^p)^q} + e g_i d_{ij} \frac{R}{N} x_{ij}^k + \right. \\
& \left. K(g_i - g_j) \right] \quad (50)
\end{aligned}$$

$$\begin{aligned}
K &= \begin{cases} (B^k - e d_{ij} \frac{R}{N}) x_{ij}^k & \text{if } g_i < g_j, \\ 0 & \text{otherwise} \end{cases} \quad (51) \\
\text{s.t. for each } k \in \{1, \dots, N\} : &
\end{aligned}$$

$$\sum_{j \in O(i)} x_{ij}^k - \sum_{j \in I(i)} x_{ji}^k = b_i \quad (52)$$

$$\begin{aligned}
b_1 &= 1, b_n = -1, b_i = 0 \quad \text{for } i \neq 1, n \\
x_{ij}^k &\in \{0, 1\} \quad (53)
\end{aligned}$$

For simplicity, we let $v_f = 1$ mile/min, $R = 1$ vehicle/min, $p = 2$, $q = 2$ and $e = 1$. The network topology used is that of Fig.2, with the distance of each arc as shown. Tab. III

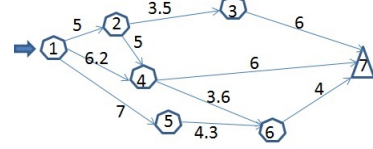


Fig. 2. A 7-node network with inhomogeneous charging nodes.

TABLE III
NUMERICAL RESULTS FOR SAMPLE PROBLEM

	P4	P5
N	2	2
obj	34.459444	35.0444
routes	1 → 5 → 6 → 7 1 → 4 → 7	1 → 5 → 6 → 7 1 → 4 → 7
N	6	6
obj	29.228750	29.6187
routes	1 → 2 → 3 → 7 (×2) 1 → 4 → 7 (×2) 1 → 5 → 6 → 7 (×2)	1 → 2 → 3 → 7 (×2) 1 → 4 → 7 (×2) 1 → 5 → 6 → 7 (×2)
N	10	10
obj	29.290898	29.6419
routes	1 → 2 → 3 → 7 (×3) 1 → 4 → 7 (×4) 1 → 5 → 6 → 7 (×3)	1 → 2 → 3 → 7 (×3) 1 → 4 → 7 (×4) 1 → 5 → 6 → 7 (×3)
N	18	18
obj		29.542209
routes	Solver Error	1 → 2 → 3 → 7 (×5) 1 → 4 → 7 (×7) 1 → 5 → 6 → 7 (×6)
N	30	30
obj		29.501931
routes	Solver Error	1 → 2 → 3 → 7 (×9) 1 → 4 → 7 (×11) 1 → 5 → 6 → 7 (×10)

shows both optimal routes and locally optimal routes obtained by solving **P4** and **P5** respectively for different values of $N \in [1, \dots, 30]$ and $G = [1 \ 1 \ 1 \ 1 \ 0.1 \ 1]$. We observe that vehicles are mainly distributed through three routes and the traffic congestion effect makes the flow distribution differ from the shortest path. The number of decision variables (hence, the solution search space) rapidly increases with the number of subflows. However, looking at Fig. 3 which gives the performance in terms of our objective functions in (30) and (50) as a function of the number of subflows, observe that the optimal objective value (**P4**) quickly converges around $N = 8$. Thus, even though the best solution is found when $N = 11$, a near-optimal solution can be determined under a small number of subflows. This suggests that one can rapidly approximate the asymptotic solution of the multi-vehicle problem (dealing

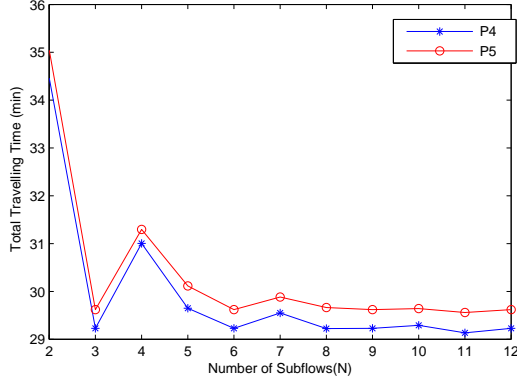


Fig. 3. Performance as a function of N (No. of subflows)

with individual vehicles routed so as to optimize a system-wide objective) based on a relatively small value of N . Another observation is that although **P5** is a suboptimal formulation it results in the same paths as those obtained by solving **P4**. Moreover, the cost difference is small over different N .

Next, we obtain a solution to the same problem (50) using the NLP formulation (48) with $0 \leq x_{ij}^k \leq 1$. Since in this example all subflows are identical, we can further combine all x_{ij}^k over each (i, j) , leading to the N -subflow relaxed problem:

$$\min_{\substack{x_{ij}^k \\ i, j \in \mathcal{N}}} \sum_{i=1}^n \sum_{j=1}^n \left[\frac{d_{ij} x_{ij} R}{v_f (1 - (x_{ij})^p)^q} + e g_i d_{ij} R x_{ij} + K(g_i - g_j) \right] \quad (54)$$

$$K = \begin{cases} (B - e d_{ij} R) x_{ij} & \text{if } g_i < g_j, \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} \text{s.t.} \quad & \sum_{j \in O(i)} x_{ij} - \sum_{j \in I(i)} x_{ji} = b_i, \quad \text{for each } i \in \mathcal{N} \\ & b_1 = 1, b_n = -1, b_i = 0, \text{ for } i \neq 1, n \\ & 0 \leq x_{ij} \leq 1 \end{aligned}$$

This is a relatively easy to solve NLP problem. Using the same parameter settings as before, we obtain the objective value of 28.5645 mins and the optimal routes are: 35.938% of vehicle flow: $(1 \rightarrow 4 \rightarrow 7)$; 28.605% of vehicle flow: $(1 \rightarrow 2 \rightarrow 3 \rightarrow 7)$; 35.457% of vehicle flow: $(1 \rightarrow 5 \rightarrow 6 \rightarrow 7)$. Compared to the best solution ($N = 11$) in Fig. 3, the difference in objective values between the integer and flow-based solutions is less than 2%. This supports the effectiveness of a solution based on a limited number of subflows in the MINLP problem.

CPU time Comparison. Tab. IV compares the computational effort in terms of CPU time for problems **P3**, **P5** and the flow control formulation to find optimal routes for the sample network shown in Fig. 2. Our results show that the flow control formulation results in a reduction of about 4 orders of magnitude in CPU time with approximately the same objective function value.

IV. CONCLUSIONS AND FUTURE WORK

We have studied the problem of minimizing the total elapsed time for energy-constrained vehicles to reach their destina-

TABLE IV
CPU TIME FOR SAMPLE PROBLEM

Fig.2 Net.	P3	P5	NLP approx.
N	3(near opt)	3(near opt)	-
obj	29.2287	29.6187	28.5645
CPU time(sec)	17122.644	190.7711	1.4

tions, including recharging when there is no adequate energy for the entire journey. In contrast to our earlier work [1], we have considered inhomogeneous charging rates at nodes. For a single vehicle, we have shown how to reduce the complexity of this problem. For a multi-vehicle problem, where traffic congestion effects are considered, we used a similar approach by aggregating vehicles into subflows and seeking optimal routing decisions for each such subflow. We also developed an alternative flow-based formulation which yields approximate solutions with a computational cost reduction of several orders of magnitude, so they can be used in problems of large dimensionality. Numerical examples show these solutions to be near-optimal. We have also found that a low number of subflows is adequate to obtain convergence to near-optimal solutions.

REFERENCES

- [1] R. Wang, C. Cassandras, and S. Pourazarm, "Energy-aware vehicle routing in networks with charging stations," *To appear in Proc. of 2014 IFAC World Congress* - arXiv:1401.6478.
- [2] G. Laporte, "The vehicle routing problem: An overview of exact and approximate algorithms," *Europ. J. Oper. Research*, vol. 59, 1992.
- [3] A. Artmeier, J. Haselmayr, M. Leucker, and M. Sachenbacher, "The optimal routing problem in the context of battery-powered electric vehicles," in *Workshop: CROCS at CPAIOR-10, 2nd International Workshop on Constraint Reasoning and Optimization for Computational Sustainability*, Bologna, Italy, May 2010.
- [4] J. Eisner, S. Funke, and S. Storandt, "Optimal route planning for electric vehicles in large networks," in *Proceedings of the 25th AAAI Conference on Artificial Intelligence*, San Francisco, US, Aug. 2011.
- [5] U. F. Siddiqi, Y. Shiraishi, and S. M. Sait, "Multi-constrained route optimization for electric vehicles (evs) using particle swarm optimization," in *2011 11th International Conference on Intelligent Systems Design and Application (ISDA)*, Cordoba, Spain, Nov. 2011, pp. 391–396.
- [6] S. Khuller, A. Malekian, and J. Mestre, "To fill or not to fill: The gas station problem," *ACM Transactions on Algorithms*, vol. 7, 2011.
- [7] M. Schneider, A. Stenger, and D. Goeke, "The electric vehicle routing problem with time windows and recharging stations," Tech Report, Dept. of Business Information Systems and Operations Research, University of Kaiserslautern, www.wiwi.de/publikationen/TheElectricVehicleRoutingProbl4278.pdf, 2012.
- [8] K. Sunder and S. Rathinam, "Route planning algorithms for unmanned aerial vehicles with refueling constraints," in *2012 American Control Conference (ACC)*, Montreal, Canada, Nov. 2012, pp. 3266–3271.
- [9] G. Joos, M. de Freige, and M. Dubois, "Design and simulation of a fast charging station for phev/ev batteries," in *Electric Power and Energy Conference (EPEC)*, 2010 IEEE, Aug. 2010.
- [10] S. Bai, Y. Du, and S. Lukic, "Optimum design of an ev/phev charging station with dc bus and storage system," in *Energy Conversion Congress and Exposition (ECCE)*, 2010 IEEE, Sep. 2010.
- [11] SAE Electric Vehicle and Plug in Hybrid Electric Vehicle Conductive Charge Coupler. SAE std. J1772, 2012.
- [12] F. S. Hillier and G. J. Lieberman, *Introduction to operations research eighth edition*. McGraw-Hill, 2005.
- [13] F. Ho and P. Ioannou, "Traffic flow modeling and control using artificial neural networks," *Control Systems, IEEE*, vol. 16, pp. 16 – 26, 1996.